JOURNAL OF APPROXIMATION THEORY 39, 153–156 (1983)

Approximation of Fourier–Laguerre Expansion by Its Cesàro Mean in Certain Banach Spaces

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Communicated by Oved Shisha

Received March 15, 1978; revised December 28, 1982

1. INTRODUCTION

Let $L[0, \infty)$ be the linear space of all complex Lebesgue measurable functions defined over the interval $[0, \infty)$. We consider the Lipschitz space Lip p consisting of all such $f \in L[0, \infty)$ which for some constant $k = k(f)$ satisfy

$$|f(x) - f(y)| \leqslant k\,|x-y|^p, \qquad 0 < p \leqslant 1 \tag{1.1}$$

Also let, for such an f,

$$\omega_\delta = \sup\{|f(x) - f(y)| : |x-y| \leqslant \delta,\ x, y \in I;\ I = [0, h],\ h < \infty\}; \tag{1.2}$$

then

$$\|f\| = |f(0)| + \sup\{\delta^{-p}\omega_\delta(f) : \delta > 0\} \tag{1.3}$$

is finite. Under this norm, Lip p is a Banach space [2, Chap. I, Exercise 22]. We denote this space by $B_L(p)$. It has as a closed subspace

$$B_l(p) = \{f \in B_L(p) : \lim_{\delta \to 0} \delta^{-p}\omega_\delta(f) = 0\}. \tag{1.4}$$

If $f \in B_l(p)$, then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(x) - f(y)| \leqslant \varepsilon\,|x-y|^p \qquad \text{for} \quad |x-y| \leqslant \delta, \quad x \in I, \quad y \in I. \tag{1.5}$$

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0021-9045/83 $3.00

The Fourier–Laguerre expansion associated with the function $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \qquad \alpha > -1, \tag{1.6}$$

where

$$\Gamma(\alpha+1) A_n^\alpha a_n = \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(\alpha)}(y)\, dy; \qquad A_n^\alpha \equiv \binom{n+\alpha}{n}. \tag{1.7}$$

$L_n^{(\alpha)}(y)$ is the nth Laguerre polynomial of order α. Here existence of the integral in (1.7) is tacitly understood. We approximate the function $f(x)$ at $x = 0$ by the Cesàro mean defined as

$$\sigma_n^k(x) = (A_n^k)^{-1} \sum_{i=0}^{n} A_{n-i}^{k-1} s_i(x), \qquad k > -1, \tag{1.8}$$

where $s_i(x)$ is the ith partial sum of the series (1.6) at the point x. Also, $\sigma_n^k(x)$ is called the nth Cesàro mean of order k of the series (1.6).

A summability theorem for the series (1.6) at $x = 0$ was given by Szegö [3]. Recently Gupta [1] proved some interesting results concerning (C, k)-summability of (1.6) at $x = 0$. These results were proven for functions with certain restrictions on integral orders. It is conceivable that improvements in the degree of convergence of the sequence of Cesàro means are possible for suitable subspaces of functions f. Our study concerns the spaces $B_l(p)$ and $B_l(p)$ $(0 < p \leqslant 1)$. We prove the following:

THEOREM. *Let* $\alpha - \frac{1}{2} < k < \alpha + \frac{1}{2}, \alpha > -1, k > -1$. *If* $f \in B_l(p)$, *where* $0 < p < 1$, *and if*

$$\begin{aligned} &\int_h^n e^{-t/2} t^{(\alpha - k - (3/2))/2} |f(t)|\, dt = o(n^{-p/2}), \\ &\int_n^\infty e^{-t/2} t^{(\alpha - k - (7/6))/2} |f(t)|\, dt = o(n^{-p/2}). \end{aligned} \tag{1.9}$$

then, as $n \to \infty$,

$$\sigma_n^k(0) = f(0) + o\{n^{(\alpha + (1/2) - k - p)/2}\}. \tag{1.10}$$

If $f \in B_L(p)$, *where* $0 < p \leqslant 1$, *and if each integral in* (1.9) *is* $O(n^{-p/2})$, *then, as* $n \to \infty$,

$$\sigma_n^k(0) = f(0) + O\{n^{1/2(\alpha + (1/2) - k - p)}\}. \tag{1.11}$$

2. Proof of the Theorem

Let $s_n(x)$ be nth partial sum of series (1.6) at any point $x \in [0, h]$ so that

$$s_n(x) = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-t} t^{\alpha} f(t)\, K_n^{(\alpha)}(x, t)\, dt, \tag{2.1}$$

where

$$K_n^{(\alpha)}(x, t) = \sum_{\nu=0}^{n} \binom{\nu+\alpha}{\nu}^{-1} L_\nu^{(\alpha)}(x)\, L_\nu^{(\alpha)}(t).$$

Since $L_\nu^{(\alpha)}(0) = \binom{\nu+\alpha}{\nu}$ and $\sum_{\nu=0}^{n} L_\nu^{(\alpha)}(t) = L_n^{(\alpha+1)}(t)$, we have

$$s_n(0) = \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} e^{-t} t^{\alpha} L_n^{(\alpha+1)}(t) f(t)\, dt. \tag{2.2}$$

Let k be any real number >-1. Then, the nth Cesàro mean of order k of the series (1.6) at $x = 0$ is given by

$$\sigma_n^k(0) = \int_0^{\infty} e^{-t} t^{\alpha} f(t) (\Gamma(\alpha+1)\, A_n^k)^{-1} \sum_{\nu=0}^{n} A_{n-\nu}^{k-1} L_\nu^{(\alpha+1)}(t)\, dt.$$

In view of the orthogonal property of Laguerre polynomials and the relation

$$(1-r)^{k-1} \sum_{n=0}^{\infty} r^n L_n^{(\alpha)}(t) = (1-r)^{k-1}\, (1-r)^{-\alpha-1} \exp\left(\frac{-rt}{1-r}\right),$$

we get

$$\sigma_n^k(0) - f(0) = (\Gamma(\alpha+1)\, A_n^k)^{-1} \int_0^{\infty} e^{-t} t^{\alpha} \{f(t) - f(0)\}\, L^{(\alpha+k+1)}(t)\, dt. \tag{2.3}$$

The theorem follows from the last equality upon estimating the integral using results of [4, Theorems 7.6.4, 8.22.1, and 8.22.5]. We omit details.

Acknowledgments

I am grateful to Professor G. S. Pandey, Ujjain, India, and to Professor O. Shisha, University of Rhode Island, for their helpful suggestions.

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